

# *Study on Dynamics of $N$ Level System of Atom by Laser Fields*

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## **Abstract**

This paper is an extension of Fujii et al (quant-ph/0307066) and in this one we again treat a model of atom with  $n$  energy levels interacting with  $n(n-1)/2$  external laser fields, which is a natural extension of usual two level system. Then the rotating wave approximation (RWA) is assumed from the beginning.

To solve the Schrödinger equation we set the **consistency condition** in our terminology and reduce it to a matrix equation with symmetric matrix  $Q$  consisting of coupling constants.

However, to calculate  $\exp(-itQ)$  **explicitly** is not easy. In the case of three and four level systems we determine it in a complete manner, so our model in these levels becomes realistic.

In last, we make a comment on Cavity QED quantum computation based on three energy levels of atoms as a forthcoming target.

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# 1 Introduction

The purpose of this paper is to develop a useful method applicable to Quantum Computation. As an easy introduction to it see for example [1], [2], [3].

Quantum Computation is in a usual understanding based on qubits which are based on two level system (two energy levels or fundamental spins of atoms), See [4], [5], [6] as for general theory of two level system.

In a realistic image of Quantum Computer we need at least one hundred atoms. However, we meet a very severe problem called **Decoherence** which may destroy a superposition of quantum states in the process of unitary evolution of the system. See for example [7] or recent [8] as an introduction. At the present time it is not easy to control Decoherence.

By the way, an atom has in general infinitely many energy levels, while in a qubit method only two energy levels are used, see for example [8] as an introduction of two level approximation. We should use this possibility to reduce a number of atoms. Since it is not realistic to take all energy levels into consideration at the same time we use  $n$  energy levels from the ground state, which is in general called qudit theory. See for example [11], [12], [13], [14] and [15].

In the paper [16] we considered a model of atom with  $n$  energy levels interacting with  $n(n-1)/2$  external laser fields, which is a natural extension of usual two level system. The rotating wave approximation (RWA) is assumed from the beginning.

In the model it is assumed that all coupling constants regarding interactions of atom with different laser fields are equal, which is nothing but an approximation theory. To construct a realistic model we again treat a “full” model of atom with  $n$  energy levels interacting with  $n(n-1)/2$  laser fields.

To solve the Schrödinger equation we set the **consistency condition** in our terminology and reduce it to a matrix equation with symmetric real matrix  $Q$  consisting of all coupling constants. However, it is not easy to calculate  $\exp(-itQ)$  **explicitly**.

Therefore we restrict to special cases. In the case of three and four level systems we determine it in a complete manner, so they become realistic. See also [17].

In last, we make a comment on Cavity QED quantum computation based on three energy

levels of atoms that is our forthcoming target.

## 2 Review on Two Level System

Let us review a two level system within our necessity and its Rabi oscillation. Let  $\{\sigma_1, \sigma_2, \sigma_3\}$  be famous Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1)$$

and we set

$$\sigma_+ \equiv \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- \equiv \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let us consider an atom with two energy levels  $E_0$  and  $E_1$  ( $E_1 > E_0$ ) as two level approximation. Its Hamiltonian is in the diagonal form given as

$$H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}. \quad (2)$$

This is rewritten as

$$H_0 = E_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (E_1 - E_0) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = E_0 \mathbf{1}_2 + \frac{\Delta}{2} (\mathbf{1}_2 - \sigma_3),$$

where  $\Delta = E_1 - E_0$  is the energy difference. Since we usually take no interest in constant terms, we can set

$$H_0 = \frac{\Delta}{2} (\mathbf{1}_2 - \sigma_3). \quad (3)$$

We consider an atom with two energy levels which interacts with external (periodic) field with  $g \cos(\omega t + \phi)$ . In the following we set  $\hbar = 1$  for simplicity. The Hamiltonian in the dipole approximation is given by

$$H = H_0 + g \cos(\omega t + \phi) \sigma_1 = \frac{\Delta}{2} (\mathbf{1}_2 - \sigma_3) + g \cos(\omega t + \phi) \sigma_1, \quad (4)$$

where  $\omega$  is the frequency of the external field,  $g$  the coupling constant between the external field and the atom. This model is complicated enough to solve, see [9], [10], [18], [19].

In the following we set  $\phi = 0$  for simplicity and assume the rotating wave approximation (which neglects the fast oscillating terms), namely

$$\cos(\omega t) = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t}) = \frac{1}{2}e^{i\omega t}(1 + e^{-2i\omega t}) \approx \frac{1}{2}e^{i\omega t},$$

and

$$\cos(\omega t)\sigma_1 = \begin{pmatrix} 0 & \cos(\omega t) \\ \cos(\omega t) & 0 \end{pmatrix} \approx \frac{1}{2} \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix},$$

therefore the Hamiltonian is given by

$$H = \frac{\Delta}{2}(\mathbf{1}_2 - \sigma_3) + \frac{g}{2}(e^{i\omega t}\sigma_+ + e^{-i\omega t}\sigma_-) \equiv \frac{\Delta}{2}(\mathbf{1}_2 - \sigma_3) + g(e^{i\omega t}\sigma_+ + e^{-i\omega t}\sigma_-) \quad (5)$$

by the redefinition of  $g$  ( $g/2 \rightarrow g$ ). It is explicitly

$$H = \begin{pmatrix} 0 & ge^{i\omega t} \\ ge^{-i\omega t} & \Delta \end{pmatrix}. \quad (6)$$

We would like to solve the Schrödinger equation

$$i\frac{d}{dt}\Psi = H\Psi. \quad (7)$$

For that purpose let us decompose  $H$  in (6) into

$$\begin{pmatrix} 0 & ge^{i\omega t} \\ ge^{-i\omega t} & \Delta \end{pmatrix} = \begin{pmatrix} 1 & \\ & e^{-i\omega t} \end{pmatrix} \begin{pmatrix} 0 & g \\ g & \Delta \end{pmatrix} \begin{pmatrix} 1 & \\ & e^{i\omega t} \end{pmatrix}, \quad (8)$$

so if we set

$$\Phi = \begin{pmatrix} 1 & \\ & e^{i\omega t} \end{pmatrix} \Psi \iff \Psi = \begin{pmatrix} 1 & \\ & e^{-i\omega t} \end{pmatrix} \Phi \quad (9)$$

then it is not difficult to see

$$i\frac{d}{dt}\Phi = \begin{pmatrix} 0 & g \\ g & \Delta - \omega \end{pmatrix} \Phi, \quad (10)$$

which is easily solved. For simplicity we set the resonance condition

$$\Delta = \omega, \quad (11)$$

then the solution of (10) is

$$\Phi(t) = \exp \left\{ -igt \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \Phi(0) = \begin{pmatrix} \cos(gt) & -i \sin(gt) \\ -i \sin(gt) & \cos(gt) \end{pmatrix} \Phi(0).$$

As a result, the solution of the equation (7) is given as

$$\Psi(t) = \begin{pmatrix} 1 \\ e^{-i\omega t} \end{pmatrix} \Phi(t) = \begin{pmatrix} 1 \\ e^{-i\omega t} \end{pmatrix} \begin{pmatrix} \cos(gt) & -i \sin(gt) \\ -i \sin(gt) & \cos(gt) \end{pmatrix} \Phi(0) \quad (12)$$

by (9). If we choose  $\Phi(0) = (1, 0)^T$  as an initial condition, then

$$\Psi(t) = \begin{pmatrix} \cos(gt) \\ -ie^{-i\omega t} \sin(gt) \end{pmatrix}. \quad (13)$$

This is a well-known model of the Rabi oscillation (or coherent oscillation).

### 3 General Theory of N Level System

In general, an atom has an infinitely many energy levels, however it is not realistic to consider all of them at the same time. Therefore we take only  $n$  energy levels from the ground state into consideration ( $E_{n-1} > \dots > E_1 > E_0$ ). Then from the lesson in the preceding section the energy Hamiltonian can be written as

$$H_0 = \begin{pmatrix} 0 & & & & \\ & \Delta_1 & & & \\ & & \Delta_2 & & \\ & & & \ddots & \\ & & & & \Delta_{n-1} \end{pmatrix}, \quad (14)$$

where  $\Delta_j = E_j - E_0$  for  $1 \leq j \leq n-1$ . For this atom we consider the interaction with  $n(n-1)/2$  independent external laser fields corresponding to every energy difference ( $E_j - E_i$  for  $j > i$ ). For example see the following figure for the case of  $n = 4$  :

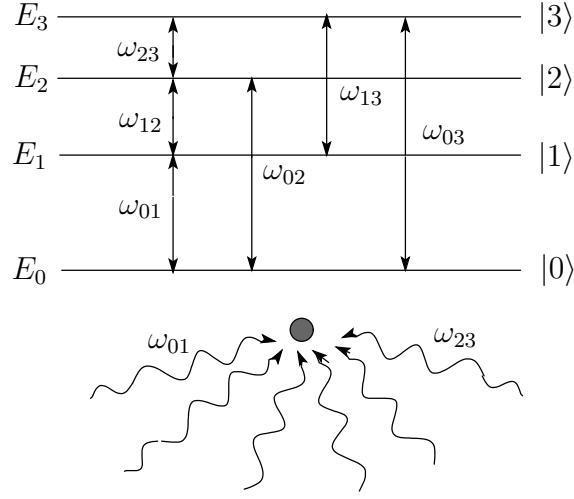


Fig.1 Atom with four energy levels and general action by laser fields

Then the interaction term assuming the RWA from the beginning is given as

$$V = \begin{pmatrix} 0 & g_{01}e^{i\omega_{01}t} & g_{02}e^{i\omega_{02}t} & \dots & \cdot & g_{0,n-1}e^{i\omega_{0,n-1}t} \\ g_{01}e^{-i\omega_{01}t} & 0 & g_{12}e^{i\omega_{12}t} & \dots & \cdot & g_{1,n-1}e^{i\omega_{1,n-1}t} \\ g_{02}e^{-i\omega_{02}t} & g_{12}e^{-i\omega_{12}t} & 0 & \dots & \cdot & g_{2,n-1}e^{i\omega_{2,n-1}t} \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & 0 & g_{n-2,n-1}e^{i\omega_{n-2,n-1}t} \\ g_{0,n-1}e^{-i\omega_{0,n-1}t} & g_{1,n-1}e^{-i\omega_{1,n-1}t} & g_{2,n-1}e^{-i\omega_{2,n-1}t} & \dots & g_{n-2,n-1}e^{-i\omega_{n-2,n-1}t} & 0 \end{pmatrix}, \quad (15)$$

where  $\{g_{\alpha\beta}\}$  are coupling constants, so the Hamiltonian is

$$H = H_0 + V. \quad (16)$$

Here we set

$$\omega_1 = \omega_{01}, \omega_2 = \omega_{12}, \dots, \omega_{n-2} = \omega_{n-3,n-2}, \omega_{n-1} = \omega_{n-2,n-1}$$

for simplicity. We would like to solve the Schrödinger equation

$$i\frac{d}{dt}\Psi = H\Psi. \quad (17)$$

Similarly in (8) let us decompose  $H$  : For

$$U = \begin{pmatrix} 1 & & & & & \\ & e^{-i\omega_1 t} & & & & \\ & & e^{-i(\omega_1+\omega_2)t} & & & \\ & & & \ddots & & \\ & & & & e^{-i(\omega_1+\omega_2+\dots+\omega_{n-2})t} & \\ & & & & & e^{-i(\omega_1+\omega_2+\dots+\omega_{n-1})t} \end{pmatrix} \quad (18)$$

it is easy to see

$$H_U \equiv U^\dagger H U = \begin{pmatrix} 0 & g_{01} & g_{02}e^{i\epsilon_{02}t} & \cdot & \cdot & g_{0,n-2}e^{i\epsilon_{0,n-2}t} & g_{0,n-1}e^{i\epsilon_{0,n-1}t} \\ g_{01} & \Delta_1 & g_{12} & \cdot & \cdot & g_{1,n-2}e^{i\epsilon_{1,n-2}t} & g_{1,n-1}e^{i\epsilon_{1,n-1}t} \\ g_{02}e^{-i\epsilon_{02}t} & g_{12} & \Delta_2 & \cdot & \cdot & g_{2,n-2}e^{i\epsilon_{2,n-2}t} & g_{2,n-1}e^{i\epsilon_{2,n-1}t} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ g_{0,n-2}e^{-i\epsilon_{0,n-2}t} & g_{1,n-2}e^{-i\epsilon_{1,n-2}t} & g_{2,n-2}e^{-i\epsilon_{2,n-2}t} & \cdot & \cdot & \Delta_{n-2} & g_{n-2,n-1}e^{-i\epsilon_{n-2,n-1}t} \\ g_{0,n-1}e^{-i\epsilon_{0,n-1}t} & g_{1,n-1}e^{-i\epsilon_{1,n-1}t} & g_{2,n-1}e^{-i\epsilon_{2,n-1}t} & \cdot & \cdot & g_{n-2,n-1}e^{-i\epsilon_{n-2,n-1}t} & \Delta_{n-1} \end{pmatrix}, \quad (19)$$

where

$$\epsilon_{ij} = \omega_{ij} - (\omega_{i+1} + \omega_{i+2} + \dots + \omega_j)$$

for  $j - i \geq 2$ .

By setting

$$\tilde{\Psi} = U^\dagger \Psi \iff \Psi = U \tilde{\Psi}$$

it is not difficult to see

$$i \frac{d}{dt} \tilde{\Psi} = \tilde{H}_U \tilde{\Psi}, \quad (20)$$

where

$$\tilde{H}_U =$$

$$\left( \begin{array}{cccccc} 0 & g_{01} & g_{02}e^{i\epsilon_{02}t} & \cdot & \cdot & g_{0,n-2}e^{i\epsilon_{0,n-2}t} & g_{0,n-1}e^{i\epsilon_{0,n-1}t} \\ g_{01} & \Delta_1 - \omega_1 & g_{12} & \cdot & \cdot & g_{1,n-2}e^{i\epsilon_{1,n-2}t} & g_{1,n-1}e^{i\epsilon_{1,n-1}t} \\ g_{0,2}e^{-i\epsilon_{02}t} & g_{1,2} & \Delta_2 - (\omega_1 + \omega_2) & \cdot & \cdot & g_{2,n-2}e^{i\epsilon_{2,n-2}t} & g_{2,n-1}e^{i\epsilon_{2,n-1}t} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ g_{0,n-2}e^{-i\epsilon_{0,n-2}t} & g_{1,n-2}e^{-i\epsilon_{1,n-2}t} & g_{2,n-2}e^{-i\epsilon_{2,n-2}t} & \cdot & \cdot & \Delta_{n-2} - \sum_{l=1}^{n-2} \omega_l & g_{n-2,n-1}e^{-i\epsilon_{n-2,n-1}t} \\ g_{0,n-1}e^{-i\epsilon_{0,n-1}t} & g_{1,n-1}e^{-i\epsilon_{1,n-1}t} & g_{2,n-1}e^{-i\epsilon_{2,n-1}t} & \cdot & \cdot & g_{n-2,n-1}e^{-i\epsilon_{n-2,n-1}t} & \Delta_{n-1} - \sum_{l=1}^{n-1} \omega_l \end{array} \right). \quad (21)$$

At this stage we take the resonance conditions

$$\begin{aligned} \Delta_1 &= \omega_1, \quad \Delta_2 = \omega_1 + \omega_2, \quad \dots, \quad \Delta_{n-2} = \sum_{l=1}^{n-2} \omega_l, \quad \Delta_{n-1} = \sum_{l=1}^{n-1} \omega_l \\ \iff \omega_j &= E_j - E_{j-1} \quad (j = 1, 2, \dots, n-1) \end{aligned} \quad (22)$$

to make the situation simpler.

Moreover, we consider a very special case : namely, in (21) we set

$$\epsilon_{ij} = 0 \iff \omega_{ij} = \omega_{i+1} + \omega_{i+2} + \dots + \omega_j = E_j - E_i \quad (23)$$

for all  $j-i \geq 2$  (see for example the figure 1 once more). We call this a **consistency condition**.

Then  $\tilde{H}_U$  becomes a constant symmetric matrix !

$$\tilde{H}_U = \left( \begin{array}{cccccc} 0 & g_{01} & g_{02} & \cdot & \cdot & g_{0,n-2} & g_{0,n-1} \\ g_{01} & 0 & g_{12} & \cdot & \cdot & g_{1,n-2} & g_{1,n-1} \\ g_{02} & g_{12} & 0 & \cdot & \cdot & g_{2,n-2} & g_{2,n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ g_{0,n-2} & g_{1,n-2} & g_{2,n-2} & \cdot & \cdot & 0 & g_{n-2,n-1} \\ g_{0,n-1} & g_{1,n-1} & g_{2,n-1} & \cdot & \cdot & g_{n-2,n-1} & 0 \end{array} \right) \equiv Q. \quad (24)$$

It is easy to solve the equation (20) with constant  $Q$

$$i \frac{d}{dt} \tilde{\Psi} = Q \tilde{\Psi}, \quad (25)$$



whose solution is formally given by

$$\tilde{\Psi}(t) = \exp(-itQ)\tilde{\Psi}(0). \quad (26)$$

As a result we have a general solution

$$\Psi(t) = U \exp(-itQ)\Psi(0). \quad (27)$$

Therefore the problem left is to calculate  $\exp(-itQ)$  **explicitly**, which is however a very hard task. In the following, let us treat the special cases  $n = 3$  and  $n = 4$  because the formula in [20] becomes helpful in the cases.

## 4 Exact Solution in Three Level System

In this section we consider the case of  $n = 3$  and calculate  $\exp(-itQ)$  in a complete manner.

For simplicity we set  $\omega_{01} = \omega_1$ ,  $\omega_{02} = \omega_3$ ,  $\omega_{12} = \omega_2$  and  $g_{01} = g_1$ ,  $g_{02} = g_3$ ,  $g_{12} = g_2$ . See the following figure.

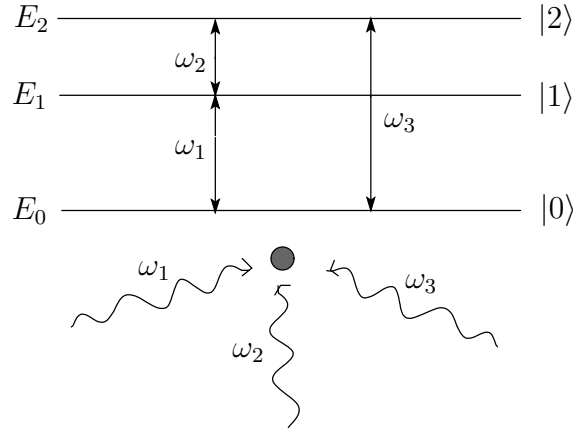


Fig.2 Atom with three energy levels and general action by laser fields

Now the matrix  $Q$  in (24) is in this case

$$Q = \begin{pmatrix} 0 & g_1 & g_3 \\ g_1 & 0 & g_2 \\ g_3 & g_2 & 0 \end{pmatrix}, \quad (28)$$

so we look for eigenvalues of  $Q$ .

From

$$0 = |\lambda \mathbf{1}_3 - Q| = \begin{vmatrix} \lambda & -g_1 & -g_3 \\ -g_1 & \lambda & -g_2 \\ -g_3 & -g_2 & \lambda \end{vmatrix} = \lambda^3 - (g_1^2 + g_2^2 + g_3^2) \lambda - 2g_1g_2g_3$$

the Cardano formula (see for example [21]) gives three real solutions

$$\lambda_1 = \alpha_+ + \alpha_-, \quad \lambda_2 = \sigma^2 \alpha_+ + \sigma \alpha_-, \quad \lambda_3 = \sigma \alpha_+ + \sigma^2 \alpha_-, \quad (29)$$

where  $\sigma = e^{2\pi i/3}$  and

$$\alpha_{\pm} = \left\{ g_1g_2g_3 \pm \sqrt{g_1^2g_2^2g_3^2 - \frac{(g_1^2 + g_2^2 + g_3^2)^3}{27}} \right\}^{\frac{1}{3}}.$$

From [20] we have finally

$$\exp(-itQ) = f_0(t)\mathbf{1}_3 + f_1(t)Q + f_2(t)Q^2 \quad (30)$$

with

$$f_0(t) = \frac{\lambda_2\lambda_3 e^{-it\lambda_1}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} + \frac{\lambda_1\lambda_3 e^{-it\lambda_2}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} + \frac{\lambda_1\lambda_2 e^{-it\lambda_3}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}, \quad (31)$$

$$f_1(t) = -\frac{(\lambda_2 + \lambda_3)e^{-it\lambda_1}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} - \frac{(\lambda_1 + \lambda_3)e^{-it\lambda_2}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} - \frac{(\lambda_1 + \lambda_2)e^{-it\lambda_3}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}, \quad (32)$$

$$f_2(t) = \frac{e^{-it\lambda_1}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} + \frac{e^{-it\lambda_2}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} + \frac{e^{-it\lambda_3}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}. \quad (33)$$

## 5 Exact Solution in Four Level System

In this section we consider the case of  $n = 4$  and calculate  $\exp(-itQ)$  in a complete manner.

For simplicity we also set  $\omega_{01} = \omega_1$ ,  $\omega_{02} = \omega_4$ ,  $\omega_{03} = \omega_6$ ,  $\omega_{12} = \omega_2$ ,  $\omega_{13} = \omega_5$ ,  $\omega_{23} = \omega_3$  and  $g_{01} = g_1$ ,  $g_{02} = g_4$ ,  $g_{03} = g_6$ ,  $g_{12} = g_2$ ,  $g_{13} = g_5$ ,  $g_{23} = g_3$ . See the figure 1 once more.

Now the matrix  $Q$  in (24) is in this case

$$Q = \begin{pmatrix} 0 & g_1 & g_4 & g_6 \\ g_1 & 0 & g_2 & g_5 \\ g_4 & g_2 & 0 & g_3 \\ g_6 & g_5 & g_3 & 0 \end{pmatrix}, \quad (34)$$

so we look for the eigenvalues of  $Q$  :

$$\begin{aligned}
0 = |\lambda \mathbf{1}_4 - Q| &= \begin{vmatrix} \lambda & -g_1 & -g_4 & -g_6 \\ -g_1 & \lambda & -g_2 & -g_5 \\ -g_4 & -g_2 & \lambda & -g_3 \\ -g_6 & -g_5 & -g_3 & \lambda \end{vmatrix} \\
&= \lambda^4 - (g_1^2 + g_2^2 + g_3^2 + g_4^2 + g_5^2 + g_6^2) \lambda^2 - 2(g_1 g_2 g_4 + g_1 g_5 g_6 + g_2 g_3 g_5 + g_3 g_4 g_6) \lambda \\
&\quad + g_1^2 g_3^2 + g_2^2 g_6^2 + g_4^2 g_5^2 - 2g_1 g_2 g_3 g_6 - 2g_1 g_3 g_4 g_5 - 2g_2 g_4 g_5 g_6 \\
&\equiv \lambda^4 + p\lambda^2 + q\lambda + r.
\end{aligned}$$

The Euler formula (see [21]) gives four real solutions

$$\lambda_1 = \alpha + \beta + \gamma, \quad \lambda_2 = \sigma^3 \alpha + \sigma^2 \beta + \sigma \gamma, \quad \lambda_3 = \sigma^2 \alpha + \beta + \sigma^2 \gamma, \quad \lambda_4 = \sigma \alpha + \sigma^2 \beta + \sigma^3 \gamma \quad (35)$$

with very complicated terms  $\alpha$ ,  $\beta$  and  $\gamma$  given below (corresponding to  $\alpha_+$ ,  $\alpha_-$  in the preceding section) and  $\sigma = e^{2\pi i/4} = i$ . For the algebraic equation with three degrees

$$64B^3 + 32pB^2 - 4(4r - p^2)B - q^2 = 0,$$

we set  $\beta = \sqrt{B}$  for the largest real solution  $B$  (this can be obtained by using the Cardano formula again).  $\alpha$  and  $\gamma$  are given as primitive solutions of equations

$$\begin{aligned}
\alpha^2 &= \frac{1}{2} \left\{ -\frac{q}{4\beta} + \sqrt{\left(\frac{q}{4\beta}\right)^2 - \left(B + \frac{p}{2}\right)^2} \right\}, \\
\gamma^2 &= \frac{1}{2} \left\{ -\frac{q}{4\beta} - \sqrt{\left(\frac{q}{4\beta}\right)^2 - \left(B + \frac{p}{2}\right)^2} \right\}.
\end{aligned}$$

From [20] we have finally

$$\exp(-itQ) = f_0(t)\mathbf{1}_4 + f_1(t)Q + f_2(t)Q^2 + f_3(t)Q^3 \quad (36)$$

with

$$f_0(t) = \frac{\lambda_2 \lambda_3 \lambda_4 e^{-it\lambda_1}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} + \frac{\lambda_1 \lambda_3 \lambda_4 e^{-it\lambda_2}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)}$$

$$+\frac{\lambda_1\lambda_2\lambda_4e^{-it\lambda_3}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)}+\frac{\lambda_1\lambda_2\lambda_3e^{-it\lambda_4}}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}, \quad (37)$$

$$f_1(t) = -\frac{(\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4)e^{-it\lambda_1}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} - \frac{(\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_3\lambda_4)e^{-it\lambda_2}}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)} \\ - \frac{(\lambda_1\lambda_2+\lambda_1\lambda_4+\lambda_2\lambda_4)e^{-it\lambda_3}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)} - \frac{(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)e^{-it\lambda_4}}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}, \quad (38)$$

$$f_2(t) = \frac{(\lambda_2+\lambda_3+\lambda_4)e^{-it\lambda_1}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} + \frac{(\lambda_1+\lambda_3+\lambda_4)e^{-it\lambda_2}}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)} \\ + \frac{(\lambda_1+\lambda_2+\lambda_4)e^{-it\lambda_3}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)} + \frac{(\lambda_1+\lambda_2+\lambda_3)e^{-it\lambda_4}}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}, \quad (39)$$

$$f_3(t) = -\frac{e^{-it\lambda_1}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} - \frac{e^{-it\lambda_2}}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)} \\ - \frac{e^{-it\lambda_3}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)} - \frac{e^{-it\lambda_4}}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}. \quad (40)$$

## 6 Approximate Solution of N Level System

Since we are interested in the whole of  $\exp(-itQ)$  in (48) we take an approximation to the equation. Namely, we assume that all coupling constants are equal :  $g = g_{\alpha\beta}$  for  $0 \leq \alpha, \beta \leq n-1$ . Then

$$Q = g \begin{pmatrix} 0 & 1 & 1 & \cdot & \cdot & 1 & 1 \\ 1 & 0 & 1 & \cdot & \cdot & 1 & 1 \\ 1 & 1 & 0 & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & 0 & 1 \\ 1 & 1 & 1 & \cdot & \cdot & 1 & 0 \end{pmatrix} \equiv gR \quad (41)$$

and it is not difficult to calculate  $\exp(-itgR)$  explicitly, [16].

If we define

$$|\mathbf{1}\rangle = (1, 1, \dots, 1, 1)^T,$$

then it is easy to see  $R = |\mathbf{1}\rangle\langle\mathbf{1}| - \mathbf{1}_n$ , so

$$\exp(-itgR) = e^{igt} \exp(-igt|\mathbf{1}\rangle\langle\mathbf{1}|).$$

Since  $\langle \mathbf{1} | \mathbf{1} \rangle = n$ ,

$$(|\mathbf{1}\rangle\langle\mathbf{1}|)^k = n^{k-1}|\mathbf{1}\rangle\langle\mathbf{1}|,$$

so that

$$\begin{aligned} \exp(-igt|\mathbf{1}\rangle\langle\mathbf{1}|) &= \mathbf{1}_n + \sum_{k=1}^{\infty} \frac{(-igt)^k}{k!} (|\mathbf{1}\rangle\langle\mathbf{1}|)^k \\ &= \mathbf{1}_n + \sum_{k=1}^{\infty} \frac{(-igt)^k}{k!} n^{k-1} |\mathbf{1}\rangle\langle\mathbf{1}| \\ &= \mathbf{1}_n + \frac{1}{n} \left\{ \sum_{k=0}^{\infty} \frac{(-ingt)^k}{k!} - 1 \right\} |\mathbf{1}\rangle\langle\mathbf{1}| \\ &= \mathbf{1}_n + \frac{\exp(-ingt) - 1}{n} |\mathbf{1}\rangle\langle\mathbf{1}|. \end{aligned} \quad (42)$$

Therefore we obtain

$$\exp(-itQ) = \exp(-itgR) = e^{igt} \left\{ \mathbf{1}_n + \frac{\exp(-ingt) - 1}{n} |\mathbf{1}\rangle\langle\mathbf{1}| \right\}. \quad (43)$$

## 7 Discussion

In this paper we developed a general theory of the  $n$  level system of atom by assuming the rotating wave approximation and reduce the Schrödinger equation to the simple matrix equation with matrix consisting of coupling constants under the **consistency condition**.

Moreover, we solved the equation in a perfect manner in the three and four level systems.

By the way, to assume the rotating wave approximation is a bit weak in the theory, so that we must study the general equation for example in the case of three level one like

$$i \frac{d}{dt} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = \begin{pmatrix} 0 & g_1 \cos(\omega_1 t + \phi_1) & g_3 \cos(\omega_3 t + \phi_3) \\ g_1 \cos(\omega_1 t + \phi_1) & \Delta_1 & g_2 \cos(\omega_2 t + \phi_2) \\ g_3 \cos(\omega_3 t + \phi_3) & g_2 \cos(\omega_2 t + \phi_2) & \Delta_2 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}. \quad (44)$$

However, it is almost impossible to solve this at the present time. Let us leave it for young researchers in Quantum Optics or Mathematical Physics as a challenging task.

Here we state our motivation once more. We would like to construct a realistic model of quantum computation with three level system. Its candidate is a quantum computation based on Cavity QED making use of three energy levels of atoms. See the following figure.

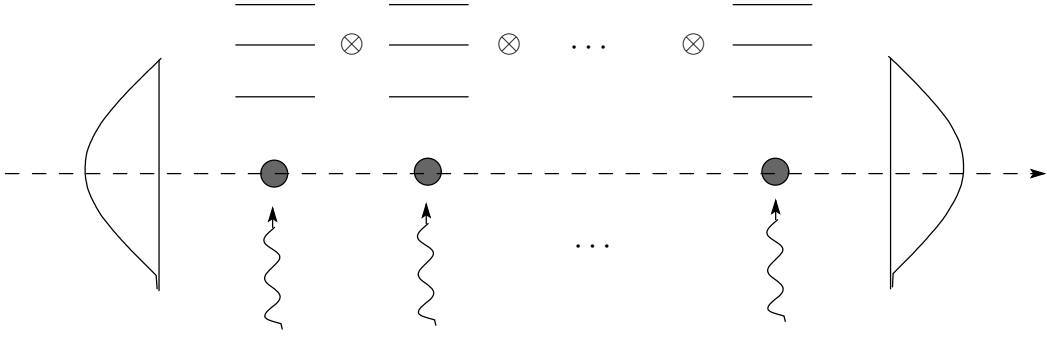


Fig.3 A general setting for a quantum computation based on Cavity QED

Let us add a short explanation to this figure. The dotted line means a single photon inserted in the cavity and all curves mean external laser fields (which are treated as classical ones) subjected to ultracold-atoms trapped linearly in it. See in detail [23] and [24] in which the Cavity QED quantum computation with two level system was “completed”.

This is our forthcoming target !

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## Appendix Diagonalization of $Q$ in the Three Level

In the case of three level system let us calculate  $\exp(-itQ)$  by use of the diagonalization method which is more popular.

In (29) we had the three eigenvalues  $\{\lambda_1, \lambda_2, \lambda_3\}$ . Therefore the normalized eigenvectors  $\{|\lambda_j\rangle \mid 1 \leq j \leq 3\}$  corresponding to these are given as

$$|\lambda_j\rangle = \begin{pmatrix} \sqrt{\frac{\lambda_j^2 - g_2^2}{3\lambda_j^2 - (g_1^2 + g_2^2 + g_3^2)}} \\ \sqrt{\frac{\lambda_j^2 - g_3^2}{3\lambda_j^2 - (g_1^2 + g_2^2 + g_3^2)}} \\ \sqrt{\frac{\lambda_j^2 - g_1^2}{3\lambda_j^2 - (g_1^2 + g_2^2 + g_3^2)}} \end{pmatrix}. \quad (45)$$

This derivation is given in the latter half. Then the matrix defined by

$$O = (|\lambda_1\rangle, |\lambda_2\rangle, |\lambda_3\rangle) \quad (46)$$

makes  $Q$  diagonal like

$$Q = O \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} O^T. \quad (47)$$

Therefore we obtain the desired form

$$\exp(-itQ) = O \begin{pmatrix} e^{-it\lambda_1} & 0 & 0 \\ 0 & e^{-it\lambda_2} & 0 \\ 0 & 0 & e^{-it\lambda_3} \end{pmatrix} O^T \equiv (a_{ij}) \quad (48)$$

where

$$a_{ij} = \sum_{k=1}^3 e^{-it\lambda_k} \sqrt{\frac{\lambda_k^2 - g_{i+1}^2}{3\lambda_k^2 - (g_1^2 + g_2^2 + g_3^2)}} \sqrt{\frac{\lambda_k^2 - g_{j+1}^2}{3\lambda_k^2 - (g_1^2 + g_2^2 + g_3^2)}}.$$

It is assumed  $g_4 \equiv g_1$  in the right hand side.

Let us calculate a special case  $g_3 = 0$ . From (29)

$$\lambda_1 = \sqrt{g_1^2 + g_2^2}, \quad \lambda_2 = 0, \quad \lambda_3 = -\sqrt{g_1^2 + g_2^2}$$

and from (45) <sup>1</sup>

$$|\sqrt{g_1^2 + g_2^2}\rangle = \begin{pmatrix} \frac{g_1}{\sqrt{2(g_1^2 + g_2^2)}} \\ \frac{1}{\sqrt{2}} \\ \frac{g_2}{\sqrt{2(g_1^2 + g_2^2)}} \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} \frac{g_2}{\sqrt{g_1^2 + g_2^2}} \\ 0 \\ \frac{-g_1}{\sqrt{g_1^2 + g_2^2}} \end{pmatrix}, \quad |-\sqrt{g_1^2 + g_2^2}\rangle = \begin{pmatrix} \frac{g_1}{\sqrt{2(g_1^2 + g_2^2)}} \\ -\frac{1}{\sqrt{2}} \\ \frac{g_2}{\sqrt{2(g_1^2 + g_2^2)}} \end{pmatrix}.$$

Therefore from

$$O = (|\sqrt{g_1^2 + g_2^2}\rangle, |0\rangle, |-\sqrt{g_1^2 + g_2^2}\rangle)$$

a straightforward calculation leads us to

$$\exp(-itQ) = \begin{pmatrix} \frac{g_1^2 \cos(t\sqrt{g_1^2 + g_2^2}) + g_2^2}{g_1^2 + g_2^2} & -i \frac{g_1 \sin(t\sqrt{g_1^2 + g_2^2})}{\sqrt{g_1^2 + g_2^2}} & \frac{g_1 g_2 \cos(t\sqrt{g_1^2 + g_2^2}) - g_1 g_2}{g_1^2 + g_2^2} \\ -i \frac{g_1 \sin(t\sqrt{g_1^2 + g_2^2})}{\sqrt{g_1^2 + g_2^2}} & \cos(t\sqrt{g_1^2 + g_2^2}) & -i \frac{g_2 \sin(t\sqrt{g_1^2 + g_2^2})}{\sqrt{g_1^2 + g_2^2}} \\ \frac{g_1 g_2 \cos(t\sqrt{g_1^2 + g_2^2}) - g_1 g_2}{g_1^2 + g_2^2} & -i \frac{g_2 \sin(t\sqrt{g_1^2 + g_2^2})}{\sqrt{g_1^2 + g_2^2}} & \frac{g_2^2 \cos(t\sqrt{g_1^2 + g_2^2}) + g_1^2}{g_1^2 + g_2^2} \end{pmatrix}. \quad (49)$$

---

<sup>1</sup>In the calculation we must choose signs of the complex roots in (45) carefully

See §3 in [17].

## Derivation of (45)

For  $Q$  in (28) we consider the equations

$$Q\mathbf{x}_j = \lambda_j\mathbf{x}_j \quad \text{for } 1 \leq j \leq 3.$$

Namely,

$$\begin{pmatrix} 0 & g_1 & g_3 \\ g_1 & 0 & g_2 \\ g_3 & g_2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda_j \begin{pmatrix} x \\ y \\ z \end{pmatrix} \iff \begin{cases} g_1y + g_3z = \lambda_jx \\ g_1x + g_2z = \lambda_jy \\ g_3x + g_2y = \lambda_jz \end{cases} \quad (50)$$

We solve the above equations with respect to  $z$ . The result is

$$x = \frac{\lambda_j g_3 + g_1 g_2}{\lambda_j^2 - g_1^2} z, \quad y = \frac{\lambda_j g_2 + g_1 g_3}{\lambda_j^2 - g_1^2} z,$$

where we have used the characteristic equation

$$\lambda_j^3 - (g_1^2 + g_2^2 + g_3^2)\lambda_j - 2g_1g_2g_3 = 0,$$

so

$$\mathbf{x}_j = z \begin{pmatrix} \frac{\lambda_j g_3 + g_1 g_2}{\lambda_j^2 - g_1^2} \\ \frac{\lambda_j g_2 + g_1 g_3}{\lambda_j^2 - g_1^2} \\ 1 \end{pmatrix}. \quad (51)$$

Next let us determine the normalization.

$$\langle \mathbf{x}_j | \mathbf{x}_j \rangle = 1 \iff 1 = x^2 + y^2 + z^2 = z^2 \frac{(\lambda_j g_3 + g_1 g_2)^2 + (\lambda_j g_2 + g_1 g_3)^2 + (\lambda_j^2 - g_1^2)^2}{(\lambda_j^2 - g_1^2)^2} \quad (52)$$

From this

$$\begin{aligned} & (\lambda_j^2 - g_1^2)^2 + (\lambda_j g_3 + g_1 g_2)^2 + (\lambda_j g_2 + g_1 g_3)^2 \\ &= \lambda_j^4 + (-2g_1^2 + g_2^2 + g_3^2)\lambda_j^2 + 4g_1g_2g_3\lambda_j + g_1^2(g_1^2 + g_2^2 + g_3^2) \\ &= \lambda_j^4 - 3g_1^2\lambda_j^2 + (g_1^2 + g_2^2 + g_3^2)\lambda_j^2 + \underline{4g_1g_2g_3}\lambda_j + g_1^2(g_1^2 + g_2^2 + g_3^2). \end{aligned} \quad (53)$$



Now we want to remove the term  $4g_1g_2g_3\lambda_j$ . By using the characteristic equation  $2g_1g_2g_3 = \lambda_j^3 - (g_1^2 + g_2^2 + g_3^2)\lambda_j$ , we have

$$\begin{aligned} \text{The right hand side of (53)} &= 3\lambda_j^4 - 3g_1^2\lambda_j^2 - (g_1^2 + g_2^2 + g_3^2)\lambda_j^2 + g_1^2(g_1^2 + g_2^2 + g_3^2) \\ &= 3\lambda_j^2(\lambda_j^2 - g_1^2) - (g_1^2 + g_2^2 + g_3^2)(\lambda_j^2 - g_1^2) \\ &= (\lambda_j^2 - g_1^2)(3\lambda_j^2 - g_1^2 - g_2^2 - g_3^2). \end{aligned} \quad (54)$$

From (52)

$$z = \frac{\sqrt{\lambda_j^2 - g_1^2}}{\sqrt{3\lambda_j^2 - g_1^2 - g_2^2 - g_3^2}},$$

so we have

$$\mathbf{x}_j = z \begin{pmatrix} \frac{\lambda_j g_3 + g_1 g_2}{\lambda_j^2 - g_1^2} \\ \frac{\lambda_j g_2 + g_1 g_3}{\lambda_j^2 - g_1^2} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3\lambda_j^2 - g_1^2 - g_2^2 - g_3^2}} \begin{pmatrix} \frac{\lambda_j g_3 + g_1 g_2}{\sqrt{\lambda_j^2 - g_1^2}} \\ \frac{\lambda_j g_2 + g_1 g_3}{\sqrt{\lambda_j^2 - g_1^2}} \\ \sqrt{\lambda_j^2 - g_1^2} \end{pmatrix} \quad (55)$$

from (51). At this stage it is easy to conjecture

$$\frac{\lambda_j g_3 + g_1 g_2}{\sqrt{\lambda_j^2 - g_1^2}} = \sqrt{\lambda_j^2 - g_2^2}, \quad \frac{\lambda_j g_2 + g_1 g_3}{\sqrt{\lambda_j^2 - g_1^2}} = \sqrt{\lambda_j^2 - g_3^2}.$$

In fact,

$$\begin{aligned} &\frac{\lambda_j g_3 + g_1 g_2}{\sqrt{\lambda_j^2 - g_1^2}} \\ &= \sqrt{\frac{(\lambda_j g_3 + g_1 g_2)^2}{\lambda_j^2 - g_1^2}} = \sqrt{\frac{g_3^2 \lambda_j^2 + 2g_1 g_2 g_3 \lambda_j + g_1^2 g_2^2}{\lambda_j^2 - g_1^2}} = \sqrt{\frac{(\lambda_j^2 - g_1^2)(\lambda_j^2 - g_2^2)}{\lambda_j^2 - g_1^2}} = \sqrt{\lambda_j^2 - g_2^2}, \end{aligned}$$

where we have used the equation

$$\begin{aligned} g_3^2 \lambda_j^2 + 2g_1 g_2 g_3 \lambda_j + g_1^2 g_2^2 &= g_3^2 \lambda_j^2 + \lambda_j^4 - (g_1^2 + g_2^2 + g_3^2)\lambda_j^2 + g_1^2 g_2^2 \\ &= \lambda_j^4 - (g_1^2 + g_2^2)\lambda_j^2 + g_1^2 g_2^2 \\ &= (\lambda_j^2 - g_1^2)(\lambda_j^2 - g_2^2). \end{aligned}$$

Similarly, we obtain

$$\frac{\lambda_j g_2 + g_1 g_3}{\sqrt{\lambda_j^2 - g_1^2}} = \sqrt{\lambda_j^2 - g_3^2}.$$

From (55) we finally obtain

$$\mathbf{x}_j = \frac{1}{\sqrt{3\lambda_j^2 - g_1^2 - g_2^2 - g_3^2}} \begin{pmatrix} \sqrt{\lambda_j^2 - g_2^2} \\ \sqrt{\lambda_j^2 - g_3^2} \\ \sqrt{\lambda_j^2 - g_1^2} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\lambda_j^2 - g_2^2}{3\lambda_j^2 - g_1^2 - g_2^2 - g_3^2}} \\ \sqrt{\frac{\lambda_j^2 - g_3^2}{3\lambda_j^2 - g_1^2 - g_2^2 - g_3^2}} \\ \sqrt{\frac{\lambda_j^2 - g_1^2}{3\lambda_j^2 - g_1^2 - g_2^2 - g_3^2}} \end{pmatrix} \equiv |\lambda_j\rangle.$$

A comment is in order.  $\sqrt{t}$  means the complex square root defined by  $\sqrt{t} = e^{i \arg t/2} \sqrt{|t|}$ , so in our case we must choose its sign (for example,  $\sqrt{4} = 2$  or  $-2$ ) carefully.

## References

- [1] H-K. Lo, S. Popescu and T. Spiller (eds) : Introduction to Quantum Computation and Information, 1998, World Scientific.
- [2] A. Barenco, C. H. Bennett, R. Cleve, D. P. Vincenzo, N. Margolus, P. Shor, T. Sleator, J. Smolin and H. Weinfurter : Elementary gates for quantum computation, Phys. Rev. A 52, 3457, 1995, quant-ph/9503016.
- [3] K. Fujii : Introduction to Grassmann Manifolds and Quantum Computation, J. Applied Math, 2(2002), 371, quant-ph/0103011.
- [4] L. Allen and J. H. Eberly : Optical Resonance and Two-Level Atoms, Wiley, New York, 1975.
- [5] P. Meystre and M. Sargent III : Elements of Quantum Optics, Springer-Verlag, 1990.
- [6] S. M. Barnett and P. M. Radmore : Methods in Theoretical Quantum Optics, Oxford University Press, 1997.
- [7] W. H. Zurek : Decoherence and the transition from quantum to classical-REVISITED, quant-ph/0306072.

- [8] M. Frasca : A modern review of the two–level approximation, *Ann. Phys.*, 306(2003), 193, quant-ph/0209056.
- [9] M. Frasca : Perturbative results on localization for a driven two–level system, *Phys. Rev. B* 68(2003) 165315, cond-mat/0303655.
- [10] K. Fujii : Two–Level System and Some Approximate Solutions in the Strong Coupling Regime, quant-ph/0301145.
- [11] K. Fujii : How To Treat An N–Level System : A Proposal, quant-ph/0302050.
- [12] K. Fujii : Exchange Gate on the Qudit Space and Fock Space, *J. Opt. B : Quantum Semiclass. Opt.*, 5(2003), S613, quant-ph/0207002.
- [13] K. Fujii : Quantum Optical Construction of Generalized Pauli and Walsh–Hadamard Matrices in Three Level Systems, quant-ph/0309132.
- [14] K. Funahashi : Explicit Construction of Controlled–U and Unitary Transformation in Two–Qudit, *Yokohama Mathematical Journal*, 52(2005), 11, quant-ph/0304078.
- [15] K. Fujii, K. Funahashi and T. Kobayashi : Jarlskog’s Parametrization of Unitary Matrices and Qudit Theory, *International Journal of Geometric Methods in Modern Physics*, 3(2006), 269, quant-ph/0508006.
- [16] K. Fujii, K. Higashida, R. Kato and Y. Wada : N Level System with RWA and Analytical Solutions Revisited, quant-ph/0307066.
- [17] K. Fujii, K. Higashida, R. Kato and Y. Wada : A Rabi Oscillation in Four and Five Level Systems, to appear in *Yokohama Mathematical Journal*, quant-ph/0312060.
- [18] J. C. A. Barata and W. F. Wreszinski : Strong Coupling Theory of Two Level Atoms in Periodic Fields, *Phys. Rev. Lett.* 84(2000), 2112, physics/9906029.

- [19] A. Santana, J. M. Gomez Llorente and V. Delgado : Semiclassical dressed states of two-level quantum systems driven by nonresonant and/or strong laser fields, J. Phys. B 34(2001), 2371, quant-ph/0011015.
- [20] K. Fujii and H. Oike : How to Calculate the Exponential of Matrices, quant-ph/0406115.
- [21] K. Fujii : A Modern Introduction to Cardano and Ferrari Formulas in the Algebraic Equations, quant-ph/0311102.
- [22] V. Ramakrishna and H. Zhou : On The Exponential of Matrices in  $su(4)$ , math-ph/0508018.
- [23] K. Fujii, K. Higashida, R. Kato and Y. Wada : Cavity QED and Quantum Computation in the Weak Coupling Regime, J. Opt. B: Quantum and Semiclass. Opt, 6(2004) 502, quant-ph/0407014.
- [24] K. Fujii, K. Higashida, R. Kato and Y. Wada : Cavity QED and Quantum Computation in the Weak Coupling Regime II : Complete Construction of the Controlled–Controlled NOT Gate, to appear in the book "Trends in Quantum Computing Research", 2006, Nova Science Publishers, Inc (USA), quant-ph/0501046.